A Study of Multielement Transmission Lines*

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Summary—**Although many papers have been published on the subject of multielement transmission lines, the application to practical problems seems rather inconvenient. The author proposes a solution to the general equations which relate the voltage difference between the lines and the mesh current. Under particular conditions, it is shown that only a single type of propagating mode exists. In this case, the solution has been obtained by the so called "decomposition method,"** ***i.e.*****, assuming several virtual two element transmission lines in lieu of the existing multielement transmission line. The problem has been solved by means of the resolved superposed virtual lines taking into account the existing boundary condition.**

INTRODUCTION

PAPERS concerning multielement transmission lines have been proposed by many authors.[1–12] However, the majority refers strictly to the general theory[3,4] and involve complicated matrix problems; the application of these results to simple practical problems is inconvenient. There are some simple problems as, for instance, the coupling theory between two wave guides,[5–7] three parallel conducting lines,[8,9] which is treated by balanced and longitudinal or by the right hand polarized, left hand polarized and longitudinal modes. However, there appears to be no paper concerning the practical calculations for the problem of more than four element transmission lines.

We shall now propose another form of the multitransmission line problem; in a particular case it can be easily solved by use of the "decomposition method," *i.e.*, by first assuming several virtual two element transmission lines concerned with the existing multielement transmission line. The problem has been solved by means of the resolved superposed virtual lines, taking into account the existing boundary conditions.

The following results were obtained:

1) Generally, n-lines have $(n-1)$ propagation modes, but in particular conditions they have only one. It is then possible to analyze most multielement transmission lines.
2) The majority of multielement line problems generally start from the fundamental linear differential equations. However, the definition of the propagation mode derived from these has not been explained in terms of the Maxwell equations.
3) As the propagation mode of the multiple line system, it may be questioned whether the TEM wave (perfect transverse wave) exists, and if it exists, whether the propagation made can be TEM. Moreover, in such a case, it is necessary to examine the relationship of the line constants.

A STUDY BASED ON ELECTROMAGNETIC WAVE THEORY

For this analysis, a perfect conductor of uniform cross section shall exist parallel in the homogeneous dielectric medium; the distance between each conductor is assumed to be extremely shortened compared to the wavelength. On the uniform line of arbitrary cross section with the axis in the z direction, an electromagnetic wave will exist in the form, $\exp j(\omega t-\beta z)$. If all the conductors are perfect conductors, the electromagnetic field E_x, H_x must satisfy $E_z=0$, $\partial H_z/\partial n=0$ on these conductors.

Accordingly $E_z\neq 0$, $H_z=0$ (TM mode), $E_z=0$ $H_z\neq 0$ (TE mode) and $E_z=H_z=0$ (TEM mode) waves on the multiline, propagate in the general shielding line; the number of these modes has no direct relation to any line numbers. It is also evident that all modes are propagated independently due to their orthogonal properties. Furthermore, when the distance between these lines can be neglected compared with the wavelength, TEM waves alone will exist, as TE and TM waves will be cut off. Under these conditions alone can the relation between the multiline equation and the Maxwell equations be found.

The general equations of multilines are given as follows:

$$\frac{dV_r}{dx} = \sum_m z_{rm} I_m, \qquad \frac{dI_r}{dx} = \sum_n y_{rn} V_n \tag{1}$$

where V_r is the potential of line r, I_r is its current and x its directional distance for its transmission line. When this system transmits the TEM wave, the field can be derived from $\exp j(\omega t\pm\beta z)$, where $\beta=\omega\sqrt{\mu\epsilon}$.

If L_{rm} is the inductance per unit length and C_{rn} is the

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capacitance z_{rm}, y_{rn} will be given by the following equations:

$$z_{rm} = j\omega L_{rm}, \; y_{rn} = j\omega C_{rn}.$$

Accordingly, from (1)

$$V_r = \pm \frac{\omega}{\beta} \sum_m L_{rm} I_m, \qquad I_r = \pm \frac{\omega}{\beta} \sum_n C_{rn} V_n. \tag{2}$$

Then, from (2)

$$V_r = \sum_m L_{rm} \sum_n C_{mn} V_n.$$

The left term in the above mentioned equation is a function for r alone. Therefore,

$$\sum_m L_{rm} C_{mr} = 1, \qquad \sum_m L_{rm} C_{mn} = 0 \quad (r \neq n)$$

or

$$\sum_m z_{rm} y_{mr} = \frac{-\omega^2}{c^2}, \qquad \sum_m z_{rm} y_{mn} = 0 \quad (r \neq n) \tag{3}$$

where c is the light velocity. As a simple example, the three-line is considered:

$$\frac{dV_1}{dx} = z_{11} I, + z_m I_2, \qquad \frac{dV_2}{dx} = z_m I_1 + z_2 I_2$$

$$\frac{dI_1}{dx} = y_1 V_1 + y_m V_2 \qquad \frac{dI_2}{dx} = y_m V_1 + y_2 V_2. \tag{4}$$

From the above equation, the following equation can be obtained.

$$\frac{d^4 V_1}{dx^4} - (Z_1 Y_1 + 2 z_m y_m + z_2 y_2) \frac{d^2 V_1}{dx^2} + (z_1 z_2 - z_m^2)(y_1 y_2 - y_m^2) V_1 = 0. \tag{5}$$

Putting $V_1 = \exp \gamma x$, the solution can be obtained by calculating γ.

$$\gamma = \pm \left[\gamma_0^2 \pm \sqrt{4\gamma_0^4 - 4(z_1 z_2 - z_m^2)(y_1 y_2 - y_m^2)}\right]^{1/2} \tag{6}$$

where

$$\gamma_0^2 = \tfrac{1}{2}(z_1 y_1 + 2 z_m y_m + z_2 y_2).$$

Substituting the relation of (3), we will obtain

$$z_1 y_1 + z_m y_m = z_2 y_2 + z_m y_m = -\frac{\omega^2}{c^2}$$

$$z_1 y_m + z_m y_2 = z_m y_1 + z_2 y_m = 0, \tag{7}$$

$$\gamma = \pm \gamma_0 = \pm \frac{j\omega}{c}. \tag{8}$$

In general multiple-element transmission lines, the matrix of the constant is written as follows:

$$[\gamma] = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots \\ \gamma_{21} & \gamma_{22} & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{bmatrix} \cdots = \tag{9}$$

The diagonal terms become $\gamma_{11} = \gamma_{22} \cdots = \gamma_0$ and the excess terms will be $\gamma_{12} = \gamma_{21} = \gamma_{13} = \cdots = 0$.

General Solution

Eq. (1) in the previous section relates the voltage and current of each line. The voltage difference of the lines is used instead of the line voltage, because it is more convenient in connection with the intrinsic transmission mode and also for handling this problem. Now, if V_{ik} is the voltage difference between the arbitrary lines i and k, as shown in Fig. 1, then

$$\frac{dV_{ik}}{dx} = \frac{dV_i}{dx} - \frac{dV_k}{dx} = \sum_{s=1}^{n} Z_{is} I_s - \sum_{s=1}^{n} Z_{ks} I_s \tag{10}$$

and the current I_s flowing on the line S is

$$\frac{dI_s}{dx} = \sum_{j=1}^{n} Y_{sj} V_{sj} \tag{11}$$

where Z_{ik} is the series impedance per unit length and Y_{ik} is the parallel admittance per unit length.

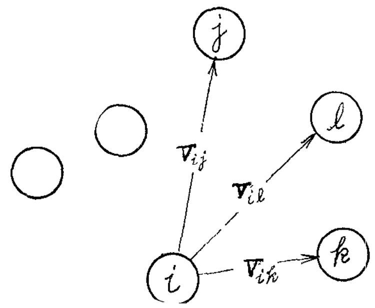

Fig. 1—Voltage difference between arbitrary lines.

From (10) and (11)

$$\frac{d^2 V_{ik}}{dx^2} = \sum_{s=1}^{n} \Gamma_{ks}{}^i V_{is} \tag{12}$$

where

$$\Gamma_{ks}{}^i = \sum_{j=1}^{n} (Z_{ij} + Z_{ks} - Z_{kj} - Z_{is}) Y_{js}. \tag{13}$$

Note: z and y in the previous section have the following relations:

$$z_{ij} = Z_{ij}, \quad y_{jk} = -Y_{jk} \; (j \neq k), \quad y_{jj} = \sum_k{}' Y_{jk}.$$

Now, in (12), we will consider the k, j component H_{kj}^{-1i} of H^{-1i} based on the specific i to set up the diagonal matrix from the similar transformation of $H^{-1}\Gamma H$:

$$V_k{}^i = \sum_j H_{kj}^{-1i} V_{ij}. \tag{14}$$

From the above transformation, we obtain

$$\frac{d^2 V_k{}^i}{dx^2} = \sum_m \left\{ \sum_{sj} H_{kj}^{-1i} \Gamma i_{js} H_{sm}{}^i \right\} = D_k{}^i V_k{}^i, \tag{15}$$

$$\therefore\ V_k{}^i = A_k \exp \sqrt{D_k{}^i}x + B_k \exp(-\sqrt{D_k{}^i}x), \quad (16)$$

$$\frac{dI_k}{dx} = \sum_k \frac{dI_{ik}}{dx} = \sum_k Y_{ik}V_{ik} \quad (17)$$

$$I_{ik} = Y_{ik}\int V_{ik}dx = \sum_{j=1}^{n} H_{kj}{}^i \frac{Y_{ik}}{\sqrt{D_k{}^i}} \tilde{V}_j{}^i, \quad (18)$$

where

$$\tilde{V}_j{}^i = A_j \exp \sqrt{D_j{}^i}x - B \exp(-\sqrt{D_j{}^i}x).$$

Thus, if we adopt a pair of $V_j{}^i$, $I_j{}^i$, the actual current I_{ik} will be shown by the intrinsic transmission mode. And the calculation of the diagonal matrix will be obtained by a similar transformation as follows.

$$|\Gamma^i - XU| = 0. \quad (19)$$

The eigenvalue D of transmission constant Γ^i based on the line i is shown by the unit matrix U and the unknown X. The K-component of characteristic column vector ϕ^{im} belonging to this root D_m is shown as follows:

$$\phi_k{}^{im} = C_m |(\Gamma^i - D_m U)^{jk}| \quad (20)$$

where C_m is an arbitrary constant and $(\Gamma^i - D_m U)^{jk}$ is the cofactor of k-component in $(\Gamma^i - D_m U)$.

Symmetrical Line

Using the above mentioned theory, we shall now carry out a simple exercise. There are three transmission lines placed parallel inside the shielding line number 4, as shown in Fig. 2.

At first, the line equation is obtained as follows from (12) and (13):

$$\frac{d^2}{dx^2}\begin{bmatrix} V_{41} \\ V_{42} \\ V_{43} \end{bmatrix} = \begin{bmatrix} \Gamma_{11}{}^4 & \Gamma_{12}{}^4 & \Gamma_{13}{}^4 \\ \Gamma_{21}{}^4 & \Gamma_{22}{}^4 & \Gamma_{23}{}^4 \\ \Gamma_{31}{}^4 & \Gamma_{32}{}^4 & \Gamma_{33}{}^4 \end{bmatrix}\begin{bmatrix} V_{41} \\ V_{42} \\ V_{43} \end{bmatrix} = [\Gamma][V]. \quad (21)$$

Each element in Γ is obtained from (13). For instance,

$$\begin{aligned} \Gamma_{11}{}^4 &= \sum_{j=1}^{4} {}^{(1)}(Z_{4j} + Z_{11} - Z_{ij} - Z_{41})Y_{j1} \\ &= (Z_{42} + Z_{11} - Z_{12} - Z_{41})Y_{21} \\ &\quad + (Z_{43} + Z_{11} + Z_{13} - Z_{41})Y_{31} \\ &\quad + (Z_{44} + Z_{11} - Z_{14} - Z_{41})Y_{41}. \end{aligned} \quad (22)$$

Other $\Gamma_{ks}{}^i$ can be acquired similarly. In a particular case, as when we consider the symmetrical line where the diagonal terms are equal and the excess terms are also equal, we obtain the following equations:

$$Z_{11} = Z_{22} = Z_{33} = Z_0, \qquad Z_{12} = Z_{23} = Z_{31} = Z_m,$$

$$Y_{41} = Y_{42} = Y_{43} = Y_0, \qquad Y_{12} = Y_{23} = Y_{31} = Y_m.$$

And, if we assume that the current flows only in the inner surface of the shielding line,

$$Z_{44} = 0.$$

Substituting these relations into $\Gamma_{ks}{}^i$

$$\Gamma_{11}{}^4 = \Gamma_{22}{}^4 = \Gamma_{33}{}^4 = Z_0Y_0 + 2(Z_0 - Z_m)Y_m$$

$$\Gamma_{12}{}^4 = \Gamma_{32}{}^4 = \Gamma_{31}{}^4 = Z_mY_0 + (Z_m - Z_0)Y_0.$$

Substituting the above mentioned equation into $[\Gamma]$ of (12),

$$[\Gamma] = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{bmatrix} \quad (23)$$

where

$$\alpha = Z_0Y_0 + 2(Z_0 - Z_m)Y_m,$$

and

$$\beta = Z_mY_0 + (Z_m + Z_0)Y_m.$$

This transmission wave has α and β mode, and because of its mutual coupling, it has complicated propagation characteristics.

We now calculate the eigenvalue of the transmission line,

$$\begin{vmatrix} \alpha - \lambda & \beta & \beta \\ \beta & \alpha - \lambda & \beta \\ \beta & \beta & \alpha - \lambda \end{vmatrix} = 0$$

and obtain the root $\alpha + 2\beta$ and the two degenerating roots $\alpha - \beta$ as follows:

$$\lambda_1 = \alpha + 2\beta = (Z_0 + 2Z_m)Y_0$$

$$\lambda_2 = \alpha - \beta = (Z_0 - Z_m)(Y_0 + 3Y_m). \quad (24)$$

It is difficult to find the line propagating the modes λ_1 and λ_2 as in the above mentioned equations, but in a particular case when it forms a symmetry, the roots can be acquired by the following method:

1) The transmission line is chosen from a pair of lines 4 and 1, 2, 3 combined as shown in Fig. 3. In this case, the admittance of a pair per unit length is $3Y_0$, the self-impedance per unit length of each line is Z_0, and the mutual impedance is $2Z_m$. The series impedance per unit length of each line is $Z_0 + 2Z_m$ as the current flows in both lines in the same direction. Therefore, the series impedance of three lines become $(Z_0 + 2Z_m/3)$ and the parallel admittance is $3Y$. Accordingly,

$$\lambda_1 = \frac{Z_0 + 2Z_m}{3} 3Y = (Z_0 + 2Z_m)Y_0.$$

This coincides with λ_1 of (24).

2) A pair of lines 2 and 3 is chosen for the transmission line, as shown in Fig. 4. In this case, lines 1 and 4 are fixed by the symmetry, and no current flows through the lines. The series impedance of lines 2 and 3 per unit length becomes $2(Z_0 - Z_m)$ as the current flows in the opposite direction.

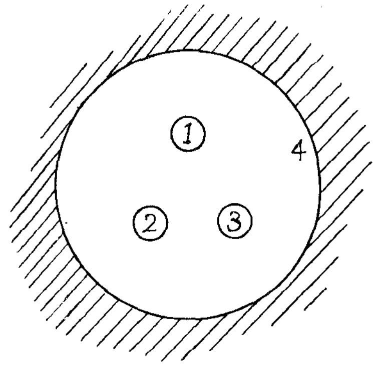

Fig. 2—Three transmission lines placed in parallel inside the shielding line number 4.

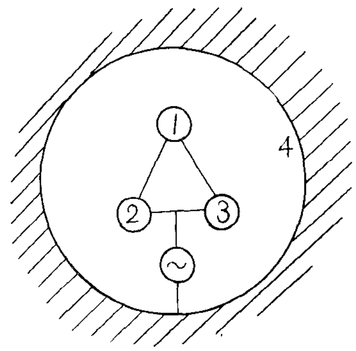

Fig. 3—A pair of lines 4 and 1, 2, 3 in a lump.

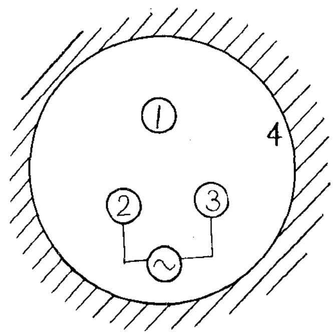

Fig. 4—A pair of lines 2 and 3.

As the parallel admittance between lines 2 and 3 is

$$Y_m + \frac{Y_0 + Y_m}{2}$$

$$\lambda_2 = 2(Z_0 - Z_m)\left(Y_m + \frac{Y_0 + Y_m}{2}\right)$$
$$= (Z_0 - Z_m)(Y_0 + 3Y_m).$$

This coincides with λ_2 in (24). As a result, the actual line can be composed by choosing two lines, 1) and 2). Now, consider the result we obtain if the arbitrary line is adopted.

3) A pair of lines 1 and 2, 3, 4 combined is chosen for the transmission line. In this case, λ_3 is calculated as follows from the same consideration:

$$\lambda_3 = \left(Z_0 - Z_m \frac{2Y_m}{Y_0 + Y_m}\right)(Y_0 + 2Y_m).$$

This value is not equal to both λ_1 and λ_2. Therefore, it is clear that finding the particular line is necessary for the intrinsic propagation. The solution of lines can be acquired from the above mentioned relations, and as these lines consist of four numbers, three equations must be prepared. For the case of 1), the voltage between line 1, 2, 3 and 4 is calculated for V_{123-4}, and for the case of 2), the voltage between line 2 and 3 is calculated for V_{2-3}, and V_{1-2} is calculated similarly.

$$\begin{aligned} V_{123-4} &= A_1 \exp\sqrt{\lambda_1}x + B_1 \exp(-\sqrt{\lambda_1}x), \\ V_{2-3} &= A_2 \exp\sqrt{\lambda_2}x + B_2 \exp(-\sqrt{\lambda_2}x), \\ V_{1-2} &= A_3 \exp\sqrt{\lambda_2}x + B_3 \exp(-\sqrt{\lambda_2}x). \end{aligned} \tag{25}$$

The line equations can be solved when arbitrary constants are determined by using the boundary condition of the actual lines derived from the above mentioned equations. The relation between the actual possible transmission lines 1) and 2) and the intrinsic propagation can obviously be obtained, as 1) and 2) are known in this example, but if the lines are unknown, the following method can be adopted.

First, we shall obtain the following equations from (20) with regard to λ_1 and λ_2.

$$\begin{bmatrix} \phi_1^{4(\lambda_1)} \\ \phi_2^{4(\lambda_1)} \\ \phi_3^{4(\lambda_1)} \end{bmatrix} = c(\lambda_1)\begin{bmatrix} \begin{vmatrix} \alpha-\lambda_1 & \beta \\ \beta & \alpha-\lambda_1 \end{vmatrix} \\ -\begin{vmatrix} \beta & \beta \\ \beta & \alpha-\lambda_1 \end{vmatrix} \\ \begin{vmatrix} \beta & \beta \\ \alpha-\lambda_1 & \beta \end{vmatrix} \end{bmatrix} = 3\beta^2 c(\lambda_1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} \phi_1^{4(\lambda_2)} \\ \phi_2^{4(\lambda_2)} \\ \phi_3^{4(\lambda_2)} \end{bmatrix} = 3\beta^2 c(\lambda_2)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ or } 3\beta^2 c(\lambda_2)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \tag{27}$$

Accordingly, we find the following relation between the voltages v_1, v_2, v_3 acting as the intrinsic propagation and the actual voltages V_{41}, V_{42}, V_{43}:

$$\begin{bmatrix} V_{41} \\ V_{42} \\ V_{43} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} v_1 &= A_1 \exp\sqrt{\lambda_1}x + B_1 \exp(-\sqrt{\lambda_1}x) \\ v_2 &= A_2 \exp\sqrt{\lambda_2}x + B_2 \exp(-\sqrt{\lambda_2}x) \\ v_3 &= A_3 \exp\sqrt{\lambda_2}x + B_3 \exp(-\sqrt{\lambda_2}x). \end{aligned}$$

Thus, each actual voltage is shown as a function of v_1, v_2 and v_3, but when each equation is written only by one intrinsic propagation mode, the voltage is chosen as follows:

$$\begin{bmatrix} V_1 \\ V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} 0 \\ -V_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} V_3 \\ -V_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}. \tag{31}$$

In this case, $V_1=v_1$, $V_2=v_2$ and $V_3=v_3$. The above mentioned equations were arranged to relate them to the existing line, and (29) becomes the function to only λ_1 and (30) and (31) are the functions to only λ_2. Namely, (29) and V_{123-4} of (25) are the same and they correspond to line 1). Similarly, (30) and V_{2-3} are the same and (31) and V_{1-2} are the same, and both equations correspond to line 2).

Decomposition Method

The analysis on symmetrical lines has been described in the previous section. Thus, if we analyze the actual voltage as the decomposite voltage V_{123-4}, V_{2-3}, V_{1-2} and so on, these will cause intrinsic propagation.

In a specific case where the transmission mode is single, a similar consideration can be applied to the usual transmission line by developing the above theory. In general, the actual voltage V_i will be given by the following equation as the linear combination of the voltage v_{ik} where the particular lines i, k are chosen for the transmission line and the line current is ceased except for the lines i, k:

$$\begin{aligned} V_i &= A_i \exp \sqrt{\lambda_0}x + B_i \exp(-\sqrt{\lambda_0}x) = \sum_k v_{ik} \\ &= \sum_k \left[a_{ik} \exp \sqrt{\lambda_0}x + b_{ik} \exp(-\sqrt{\lambda_0}x)\right]. \end{aligned} \tag{32}$$

But a_{ik} and b_{ik} should be determined by the boundary condition in the multiline and not by i k lines independently. The current in this case is shown as follows:

$$i_{ik} = Y_{0ik}\left[a_{ik} \exp \sqrt{\lambda_0}x - b_{ik} \exp(-\sqrt{\lambda_0}x)\right]. \tag{33}$$

Where Y_{0ik} is the characteristic admittance between i and k, it can be calculated by the following equation:

$$Y_{0ik} = c\, C_{ik} \tag{34}$$

where C_{ik} is the electrostatic capacity per unit length between i and k and c is the light velocity.

We can now consider the ideal equation of the n transmission line which satisfies the condition as follows. Let us first take the condition in which the current flows only in two arbitrary lines, namely the i, k elements, with no current flowing in the other elements. Since the number of independent components of voltage or current is $n-1$, we will be able to find the other if we know the $n-1$ terminal voltage or line-current.

$$\begin{aligned} v_{ik} &= a_{ik} \exp \sqrt{\lambda_0}x + b_{ik} \exp(-\sqrt{\lambda_0}x) \\ i_{ik} &= Y_{0ik}\left[a_{ik} \exp \sqrt{\lambda_0}x - b_{ik} \exp(-\sqrt{\lambda_0}x)\right] \\ &= Y_{0ik}\bar{v}_{ik}. \end{aligned} \tag{35}$$

In Fig. 5, the condition in which the current flows only between i and k is

$$\begin{aligned} i_j &= \sum_l{}' Y_{0jl}v_{jl} = \sum_l{}' Y_{0jl}(v_{ji} + v_{il}) \\ &= \sum_l{}' Y_{0jl}(x_l\bar{v}_{ik} - x_j\bar{v}_{ik}) = 0. \end{aligned}$$

Therefore,

$$\sum_l{}' Y_{0jl}(x_l - x_j) = 0 \tag{36}$$

where $\sum{}'$ shows the total summation $(1, 2, 3 \cdots, n)$ except j, assuming $x_i=0$ and $x_k=1$. Eq. (36) is $(n-2)$ linear equation concerning $(n-2)$ of the unknown numbers, $x_1, x_2, \cdots, x_n$ (except $n=i, k$). That is

$$\begin{bmatrix} -\sum_l{}' Y_{01l} & Y_{012} & \cdots & Y_{01n} \\ & Y_{021} - \sum_l{}' Y_{02l} & & \\ & \cdot & & \\ & \cdot & & \\ & Y_{0n1} & \cdots & -\sum_l{}' Y_{0nl} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} -Y_{01k} \\ -Y_{02k} \\ \cdot \\ \cdot \\ -Y_{0nk} \end{bmatrix}. \tag{37}$$

Solving the above mentioned equation, if we add a voltage x_j times the $i.k.$ voltage between i and l lines, to all the l lines with the exception of $i.k.$, the line current except i, j becomes zero. And the ith line current becomes

$$i_i = \sum_l{}' Y_{0il}\bar{v}_{il} = \left(\sum_l{}' Y_{0il}x_l\right)\bar{v}_{ik}. \tag{38}$$

We call such a line the "fundamental transmission line" with respect to the line element i, k. We now divide the n-line into m groups which are called line bundle L $(1, 2, \cdots, q)$ and those bundles preserve the same potential as shown in Fig. 6. Namely, they are to be jointed to each line with a perfect conductor line of infinitesimal small diameter. Thus, we can deal with the mth transmission line. The characteristic admittance between B $(1, 2, \cdots, p)$ and L $(1, 2, \cdots, q)$ of the arbitrary line bundle is as follows:

$$i_{Bb} = \sum_{b=1}^{p}\sum_{l=1}^{q} i_{BbDd} = \sum_{b=1}^{p}\sum_{l=1}^{q} Y_{0BbLl}\bar{v}_{BL} \tag{39}$$

$$Y_{0BL} = \sum_{b=1}^{p}\sum_{l=1}^{q} Y_{0BbLl}. \tag{40}$$

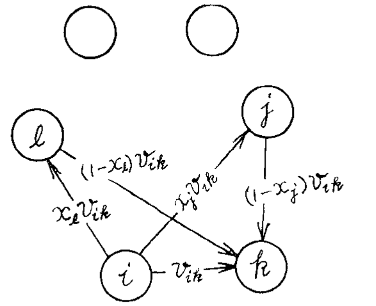

Fig. 5—Condition in which current flows only between i and k.

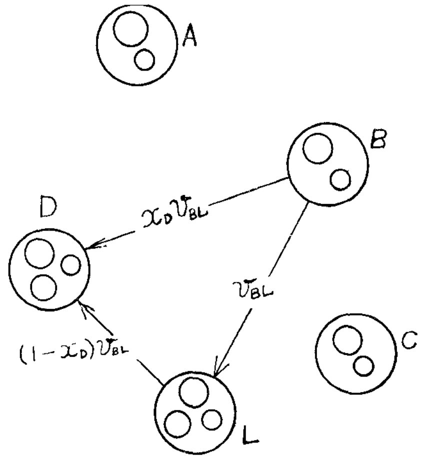

Fig. 6—Transmission line dividing some groups.

The current i_{Bb} which runs into each element Bb in the line bundle B and the current distributed factor α will be

$$
\begin{aligned}
i_{Bb} &= \sum_D{}' \sum_d i_{BbDd} = \sum_D{}' \sum_d Y_{0BbDd} x_D v_{BL} \\
&= \frac{\sum_D{}' \sum_d Y_{0BbDd} x_D}{\sum_D{}' Y_{0BD} x_D}\, i_{BL} = \frac{\sum_D{}' \sum_d Y_{0BbDd} x_D}{\sum_D{}' \sum_b \sum_d Y_{0BbDd} x_D}\, i_{BL}
\end{aligned} \tag{41}
$$

$$
i_{Bb} = \alpha i_{BL}. \tag{42}
$$

Also, the current i_{Aa} which runs into each element a in the arbitrary bundle A except BL, is derived as follows:

$$
i_{Aa} = \sum_D{}' \sum_d i_{AaDd} \frac{\sum_D{}' \sum_d Y_{0AaDd}(x_D - x_A)}{\sum_D{}' \sum_b \sum_d Y_{0BbDd} x_D}\, i_{BL}. \tag{43}
$$

When the given n transmission line is considered as a line bundle $A, B \cdots M$ divided properly into m numbers, it can be regarded as a two-transmission line; the characteristic admittance is derived by $\sum_D{}' Y_{0BD} x_D$ for any two-line BL. It may be called the fundamental transmission line on line element i, k. In n transmission line, many fundamental lines are found by various arrangements. Any n transmission line can be formed by superposition of v_{ik}, i_{ik} for a suitable fundamental transmission line $(n-1)$, as the degree of freedom of an n-line is $(n-1)$.

In this case, every line element must be used as an element of the fundamental line at least once. What $(n-1)$ fundamental transmission line should be adopted is easily decided by analyzing it properly according to the given problem.

Example: "Split Coaxial-Type Balun" as a Three-Transmission Line

Fig. 7(a) shows the equivalent circuit of the split coaxial-type balun terminating with Y_{12}, Y_{23}, Y_{31}, neglecting the earth effect. This circuit can be decomposited from two types of transmission lines, as shown in Fig. 7(b) and (c).

By (40) and (42) in the previous section, the characteristic admittance and the current distributed factor α are shown as follows:

$$
Y_{013-2} = Y_{021} + Y_{023}
$$

$$
\alpha = \frac{Y_{012}}{Y_{012} + Y_{023}}.
$$

Also, the voltage divider factor x_2 from (37) is shown as follows:

$$
-\sum_S Y_{02S} x_2 = -Y_{021}
$$

$$
x_2 = \frac{Y_{012}}{Y_{012} + Y_{023}}.
$$

Relations between the voltage and the current in Fig. 7(a) are derived by superposition in (b) and (c).

$$
\begin{aligned}
I_1 &= -\frac{Y_{012}}{Y_{012} + Y_{023}} i_1 - i_2 \\
I_2 &= i_1 \\
V_{23} &= v_1 - \frac{Y_{012}}{Y_{012} + Y_{023}} v_2 \\
V_{31} &= v_2.
\end{aligned}
$$

Applying the boundary condition at the termination $l=0$, voltage and current at this surface are shown as follows:

$$
\begin{aligned}
I_1 &= Y_{12} V_{23} - (Y_{12} + Y_{31}) V_{31} \\
I_2 &= (Y_{13} + Y_{23}) V_{23} + Y_{12} V_{31} \\
-i_2 &= -j\left(Y_{031} + \frac{Y_{032} Y_{012}}{Y_{031} + Y_{023}} \right) \cot \beta l_0 v_2 = Y_{s13} v_2 \\
i_1 &= (Y_{12} + Y_{23}) v_1 \\
&\quad + \left(Y_{12} \frac{Y_{023}}{Y_{012} + Y_{023}} - \frac{Y_{012}}{Y_{012} + Y_{023}} Y_{23} \right) v_2.
\end{aligned}
$$

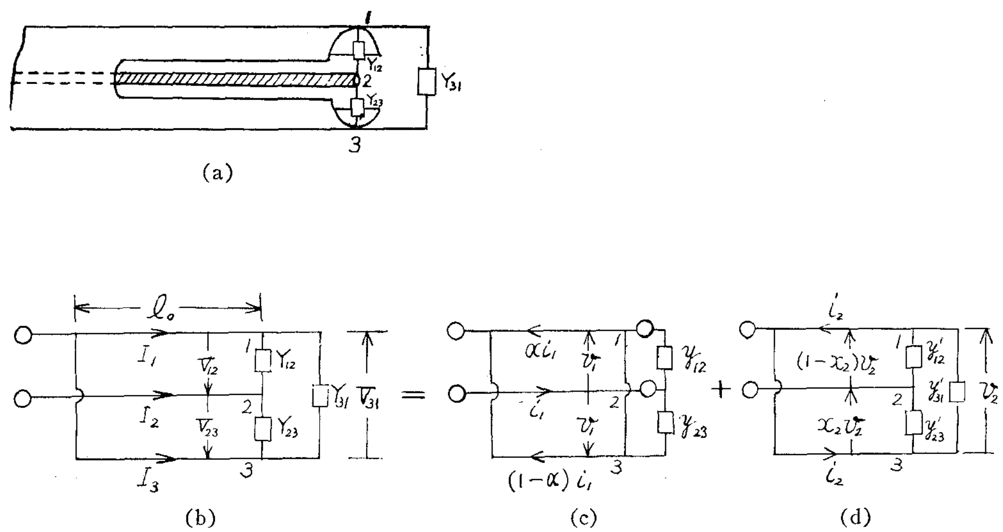

Fig. 7—Split coaxial type balun. (a) Construction. (b) Equivalent circuit. (c), (d) Decomposite circuit.

Input admittance Y_{in} is shown as follows:

$$Y_{in} = \frac{i_1}{v_1}$$

$$= \frac{(Y_{12} + Y_{23})(Y_{31} + Y_{S13}) + Y_{23}Y_{12}}{Y_{31} + Y_{s13} + Y_{12}\frac{Y_{023}^2}{(Y_{012} + Y_{023})^2} + Y_{23}\frac{Y_{012}^2}{(Y_{012} + Y_{023})^2}}.$$

In the case of symmetrical split, $Y_{012} = Y_{023}$, then

$$Y_{\text{in}} = \frac{4(Z_1 + Z_2 + Z_R)}{4Z_1Z_2 + Z_1Z_R + Z_2Z_R}$$

where

$$Z_1 = \frac{1}{Y_{12}}, \qquad Z_2 = \frac{1}{Y_{23}}, \qquad Z_R = \frac{1}{Y_{31} + Y_{s13}}.$$

When a segment of the outer split cylinder is shorted to the central conductor,

$$Y_{12} = \infty, \quad \text{then} \quad Y_{\text{in}} = 4(Y_{23} + Y_{31} + Y_{s13}),$$

putting

$$Y_{23} + Y_{31} = Y_R$$
$$Y_{\text{in}} = 4(Y_R + Y_{s13}).$$

In other words, the input admittance is equal to four times the load Y_R and slot admittance Y_{s13}.

Conclusion

The main object of this paper is to discuss transmission modes and, under particular conditions where there is only one mode in the line, the adoption of the specific solution: the decomposition method.

This analysis based on the decomposition method can be used to produce useful solutions for many complex practical problems, *i.e.*, balun, diplexer, etc.

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